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# Theory of the almost-highest wave. Part 2. Matching and analytic extension

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Most methods of calculating steep gravity waves (of less than the maximum height) encounter difficulties when the radius of curvature $R$ at the crest becomes small compared with the wavelength $L$, or some other typical length scale. This paper describes a new method of calculation valid when $R/L$ is small.

For deep-water waves, a parameter $\epsilon$ is defined as equal to $q/2^{\frac{1}{2}}c_0$, where $q$ is the particle speed at the wave crest, in a frame of reference moving with the phase speed $c$. Hence $\epsilon$ is of order $(R/L)^{\frac{1}{2}}$. Three zones are distinguished: (1) an inner zone of linear dimensions $\epsilon^2 L$ near the crest, where the flow is described by the inner solution found previously by Longuet-Higgins & Fox (1977); (2) an outer zone of dimensions $O(L)$ where the flow is given by a perturbed form of Michell's solution for the highest wave; and (3) a matching zone of width $O(\epsilon L)$. The matching procedure involves complex powers of $\epsilon$.

The resulting expression for the square of the phase velocity is found to be

$$c^2 = (g/k)\{1{\cdot}1931 - 1{\cdot}18\epsilon^3 \cos(2{\cdot}143 \ln \epsilon + 2{\cdot}22)\}$$

(see figures 5*a*, *b*), which is in remarkable agreement with independent calculations based on high-order series. In particular, the existence of turning-points in the phase velocity as a function of wave height is confirmed.

Similar expressions, valid to order $\epsilon^3$, are found for the wave height, the potential and kinetic energies and the momentum flux or impulse of the wave.

The velocity field is extended analytically across the free surface, revealing the existence of branch-points of order $\frac{1}{2}$, as predicted by Grant (1973).

## 1. Introduction

Most serious attempts to calculate the form of steep gravity waves on water of infinite depth have involved lengthy techniques, for example the summation of small amplitude expansions carried to very high order (Schwartz 1974; Longuet-Higgins 1975; Cokelet 1977) or the solution of integral equations (Milne-Thomson 1968) or other numerical methods (Yamada 1957; Sasaki & Murakami 1973) all of which become increasingly laborious as the wave of maximum height is approached.

Interest in the problem has nonetheless increased not only because of possible applications but also in view of the unexpected discovery that several overall properties

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of the wave motion – for example the momentum, energy and speed of propagation – are not steadily increasing functions of the wave amplitude at a given wavelength, but instead have maxima well within the possible range of wave steepness (Longuet-Higgins 1975; Cokelet 1977).

The present paper continues the development of a new and simpler approach to the calculation of steep gravity waves, begun in an earlier paper (Longuet-Higgins & Fox 1977, to be referred to as paper I). It was there shown that in any progressive gravity wave, whether in deep or in shallow water, the flow near the wave crest tends to a certain asymptotic form, within a distance from the crest comparable to the radius of curvature $1/\kappa$. Details of this limiting flow are given in figures 6, 7 and 9 of paper I.

In the present paper it will be shown how this flow may be matched, as an 'inner solution' valid near the crest, to an outer solution representing the flow in the rest of the wave. Two terms are sufficient to provide an accurate description of the wave, thus greatly simplifying calculations.

In §2 we first give some details of the matching procedure, in terms of a small parameter $\epsilon$ of order $q/c$, where $q$ is the particle speed at the crest, in a frame moving with the phase speed $c$. It turns out that the matching must take account of terms which are oscillatory in the physical co-ordinates, as pointed out in paper I. In §3 we derive the lowest-order outer solution for waves in deep water by a method essentially similar to Michell's (1893) but carried to greater accuracy; and in the following section (§4) we derive the lowest-order perturbation to this flow due to the presence of the rounded crest. The final matching is carried out in §5, where asymptotic expressions are found for the height of the waves, the phase speed and other quantities in terms of $\epsilon$. These simple expressions are compared with those obtained by much longer methods and are found to be in good agreement.

## 2. The matching technique

Our general method is applicable to waves in shallow water or indeed to any steady, free-surface flow with a sharply curved crest. To fix the ideas, however, consider waves in deep water as in figure 1. Define

$$\epsilon = q/2^{\frac{1}{2}}c_0, \tag{2.1}$$

where $c_0$ is the speed of waves of wavelength $L$ and infinitesimal amplitude:

$$c_0 = (gL/2\pi)^{\frac{1}{2}}. \tag{2.2}$$

It will be convenient to choose units such that

$$g = 1, \quad L = 2\pi, \quad c_0 = 1. \tag{2.3}$$

We distinguish three different zones (figure 1). The inner zone I is a region of dimensions $O(\epsilon^2)$ surrounding the wave crest. In this zone, typical velocities are of order $\epsilon$, typical lengths (such as the radius of curvature at the crest) are of order $\epsilon^2$, and the complex velocity potential $\chi$ is typically of order $\epsilon^3$. In the outer zone III all velocities, lengths and potentials are of order 1. In the intermediate zone II, whose scale is of order $\epsilon$, velocities are of order $\epsilon^{\frac{1}{2}}$ while $\chi$ is of order $\epsilon^{\frac{3}{2}}$.

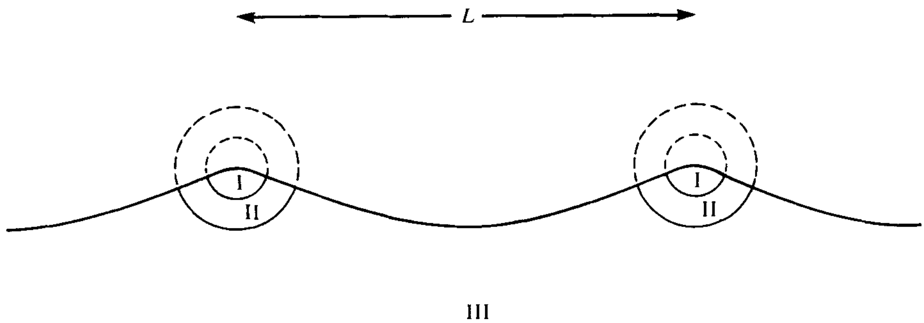



FIGURE 1. Sketch showing regions of validity for the inner solution (zone I), the outer solution (zone III) and matching (zone II).

Consider first the inner zone I. Here we may write

$$z = \epsilon^2 z_1, \quad \chi = \epsilon^3 \chi_1, \tag{2.4}$$

where $z_1$ and $\chi_1$ are of order unity. In paper I it was shown that the outer expansion of the flow in this zone, in the limit $\epsilon \to 0$, was given by

$$i\chi_1 \sim \zeta_1 - \tfrac{1}{2}(Q\zeta_1^{-i\mu} + Q^* \chi_1^{i\mu}), \tag{2.5}$$

where $\zeta_1 = \frac{2}{3}z^{\frac{3}{2}}$, $Q$ is a constant, $Q^*$ its complex conjugate, and $\mu$ is the positive real root of the equation

$$(\tfrac{1}{2}\pi\mu)\tanh(\tfrac{1}{2}\pi\mu) = \pi/(2 \times 3^{\frac{1}{2}}), \tag{2.6}$$

so that

$$\mu = 0{\cdot}7143 \ldots. \tag{2.7}$$

For large values of $\zeta_1$ (or $z_1$) equation (2.5) is clearly equivalent to

$$z_1 \sim (\tfrac{3}{2}i\chi_1)^{\frac{2}{3}} + A(i\chi_1)^{-\frac{1}{3}-i\mu} + A^*(i\chi_1)^{-\frac{1}{3}+i\mu}, \tag{2.8}$$

where $A$ is a constant. From the calculations in paper I it was found that

$$A = 0{\cdot}201 e^{-0{\cdot}47i}. \tag{2.9}$$

The first term on the right of (2.8) represents the Stokes (1880) 120° corner flow, and the remaining terms represent displacements vanishing (in an oscillatory fashion) as $|\chi_1|$ (or $|z_1|$) tends to infinity.

In the outer zone III, we know from Grant (1973) that an expansion valid for the sharp-crested wave ($\epsilon = 0$) as $z \to 0$ is

$$z \sim (\tfrac{3}{2}i\chi)^{\frac{2}{3}} + B(i\chi)^{\nu}, \tag{2.10}$$

where $B$ and $\nu$ are constants, with $\nu$ satisfying

$$\tan(\tfrac{1}{2}\pi\nu) = -(4+3\nu)/3^{\frac{3}{2}}\nu. \tag{2.11}$$

It was pointed out in paper I that by writing $\nu = -(\frac{1}{3}+\lambda)$ equation (2.11) reduces to

$$(\tfrac{1}{2}\pi\lambda)\tan(\tfrac{1}{2}\pi\lambda) = -\pi/(2 \times 3^{\frac{1}{2}}) \tag{2.12}$$

[compare with (2.6)]. We now require that $\nu > \frac{2}{3}$, i.e. $\lambda < -1$. The smallest such root is

$$\lambda = -1{\cdot}8027\ldots. \tag{2.13}$$

The constant $B$ will be determined in §3.

Now to match the solutions in zone II we write

$$z = \epsilon z_2, \quad \chi = \epsilon^{\frac{3}{2}} \chi_2, \tag{2.14}$$

where $z_2$, $\chi_2$ are to be of order 1. Since then

$$z_1 = \epsilon^{-1} z_2, \quad \chi_1 = \epsilon^{-\frac{3}{2}} \chi_2 \tag{2.15}$$

we assume that in zone II

$$z_2 \sim (\tfrac{3}{2} i \chi_2)^{\frac{2}{3}} + \frac{A \epsilon^{\frac{3}{2}+\frac{3}{2}i\mu}}{(i\chi_2)^{\frac{1}{3}+i\mu}} + \frac{A^* \epsilon^{\frac{3}{2}-\frac{3}{2}i\mu}}{(i\chi_2)^{\frac{1}{3}-i\mu}} + \frac{B \epsilon^{-\frac{3}{2}-\frac{3}{2}\lambda}}{(i\chi_2)^{\frac{1}{3}+\lambda}}. \tag{2.16}$$

In terms of $z$ and $\chi$,

$$z \sim (\tfrac{3}{2} i\chi)^{\frac{2}{3}} + B(i\chi)^{-\frac{1}{3}-\lambda} + A\epsilon^{3+3i\mu}(i\chi)^{-\frac{1}{3}-i\mu} + A^*\epsilon^{3-3i\mu}(i\chi)^{-\frac{1}{3}+i\mu} \tag{2.17}$$

as $\chi \to 0$. So the solution in zone III must be that for the sharp-crested wave, with asymptotic form (2.10), together with a first correction for non-zero $\epsilon$ with asymptotic form

$$A\epsilon^{3+3i\mu}(i\chi)^{-\frac{1}{3}-i\mu} + A^*\epsilon^{3-3i\mu}(i\chi)^{-\frac{1}{3}+i\mu} \tag{2.18}$$

as $\chi \to 0$. This correction will be calculated in §4. Similarly the solution in zone I is the flow described in paper I, with a correction for non-zero $\epsilon$ which has the asymptotic form

$$B\epsilon^{-3-3\lambda}(i\chi_1)^{-\frac{1}{3}-\lambda} \tag{2.19}$$

as $\chi_1 \to \infty$.

## 3. The outer solution: lowest order

As before, we take axes moving horizontally with the phase speed $c$, and treat $\chi = \phi + i\psi$ as the independent variable. We seek $z = x + iy$ so as to satisfy the boundary condition

$$2\,\mathrm{Re}\, z |dz/d\chi|^2 = 1 \tag{3.1}$$

on the free surface $\psi = 0$. The motion is periodic in $\phi$ with period $2\pi c$. The origin of $\chi$ being taken at a wave crest, we know that the solution must have a singularity there of the form

$$z \sim (\tfrac{3}{2} i\chi)^{\frac{2}{3}},$$

which may be more conveniently written

$$z \sim (\tfrac{3}{2}c)^{\frac{2}{3}} (1 - e^{-i\chi/c})^{\frac{2}{3}}, \tag{3.2}$$

while as $\psi \to -\infty$ so $z \sim i\chi/c$. Modifying the expansion proposed by Michell (1893), we first map the flow region in the complex potential plane

$$\psi \leqslant 0, \quad -\pi \leqslant \phi/c \leqslant \pi$$

(figure 2) into a circle in the $\omega$ plane (figure 3) by the transformation

$$\omega = \frac{(1+s)\, e^{-i\chi/c} + (1-s)}{(1-s)\, e^{-i\chi/c} + (1+s)}, \tag{3.3}$$

where $s$ is a real parameter at our disposal. As $\psi \to -\infty$, that is as

$$\omega \to (1-s)/(1+s)$$

(the point $I''$ in figure 3), we require that the flow shall tend to a uniform stream: $z - i\chi/c \to$ constant. $z - i\chi/c$ is single-valued, bounded and hence analytic in the neigh-

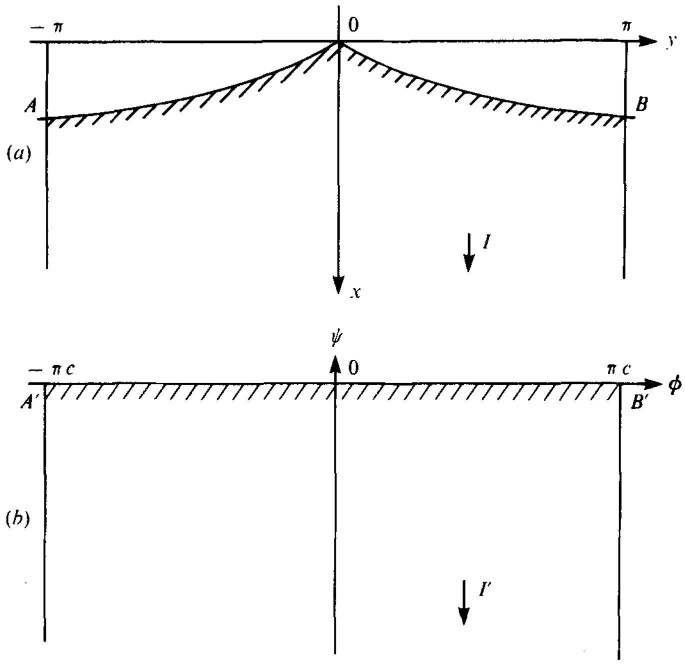



FIGURE 2. Co-ordinates (a) in the physical plane and (b) in the plane of the complex potential $\chi = \phi + i\psi$.

bourhood of $I''$. In view of (3.2) we write

$$z - i\chi/c = -(1-e^{-i\chi/c}) + (1-e^{-i\chi/c})^{\frac{2}{3}}(a_0 + a_1\omega + a_2\omega^2 + \ldots), \tag{3.4}$$

where the coefficients $a_n$ are real and the term $-(1-e^{-i\chi/c})$ has been added in order to cancel the $-i\chi/c$ term on the left-hand side to order $\chi$ near the crest, so improving the rate of convergence of the power series there. On the free surface we have $\psi = 0$, $|\omega| = 1$, and

$$2\,\mathrm{Re}\,z \doteq (1-e^{-i\chi/c})^{\frac{2}{3}}\mathscr{A} + \text{complex conjugate}, \tag{3.5}$$

where

$$\mathscr{A}(\omega) = a_0 + a_1\omega + \ldots - \left\{\frac{2(1-\omega)}{(1+s)-(1-s)\omega}\right\}^{\frac{1}{3}}.$$

On differentiating (3.4) we have

$$-ic\,dz/d\chi = (1-e^{-i\chi/c})^{-\frac{1}{3}}\{(1-s)\omega-(1+s)\}^{-1}\mathscr{B}, \tag{3.6}$$

where

$$\begin{aligned}\mathscr{B}(\omega) = {} & \{(1-s)\omega-(1+s)\}\left\{\frac{2(1-\omega)}{(1+s)-(1-s)\omega}\right\}^{\frac{4}{3}} \\ & + \tfrac{2}{3}\{(1-s)-(1+s)\omega\}(a_0+a_1\omega+\ldots) \\ & + (2s)^{-1}(1-\omega)\{(1-s)\omega-(1+s)\}\{(1-s)-(1+s)\omega\}(a_1+2a_2\omega+\ldots).\end{aligned}$$

Now substituting these expressions in the free-surface condition (3.1) we obtain

$$\begin{aligned}&\left(\frac{1-e^{-i\chi/c}}{1-e^{i\chi/c}}\right)^{\frac{1}{3}}\mathscr{A}(\omega)\,\mathscr{B}(\omega)\,\mathscr{B}(\omega^{-1}) + \text{complex conjugate} \\ &\quad = c^2\{(1-s)\omega-(1+s)\}\{(1-s)-(1+s)\omega\}.\end{aligned} \tag{3.7}$$

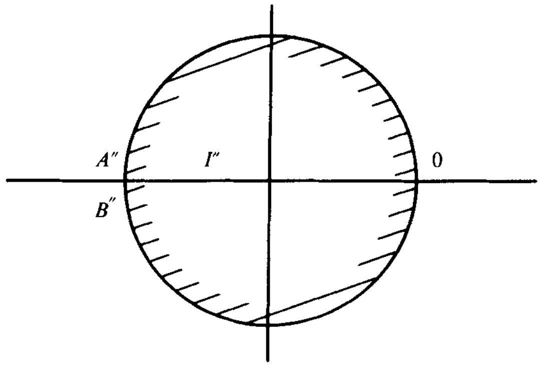



FIGURE 3. The $\omega$ plane.

| $s$ | $N$ | $c^2$ | Maximum percentage error |
|---|---|---|---|
| 2 | 20 | 1·1927 | 0·6 |
| 2 | 40 | 1·1928 | 0·5 |
| 2 | 60 | 1·1929 | 0·4 |
| 5 | 20 | 1·1929 | 0·5 |
| 5 | 40 | 1·1930 | 0·3 |
| 5 | 60 | 1·1930 | 0·3 |
| 10 | 20 | 1·1713 | 5 |
| 10 | 40 | 1·1931 | 0·3 |
| 10 | 60 | 1·1931 | 0·2 |
| 20 | 40 | 1·1701 | 5 |
| 20 | 60 | 1·1924 | 0·6 |

TABLE 1. Estimates of $c^2$ for limiting waves in deep water, with the corresponding maximum error in the free-surface condition.

Note that, since $\omega = e^{i\theta}$ on the free surface, $\mathscr{B}(\omega^{-1})$ is the complex conjugate of $\mathscr{B}(\omega)$. Now

$$\left(\frac{1-e^{-i\chi/c}}{1-e^{i\chi/c}}\right)^{\frac{1}{3}} = \left\{\frac{1-\omega}{1-\omega^{-1}}\frac{(1+s)-(1-s)\,\omega^{-1}}{(1+s)-(1-s)\,\omega}\right\}^{\frac{1}{3}}$$
$$= (-\omega)^{\frac{1}{3}}\left\{1-\frac{1-s}{1+s}\omega^{-1}\right\}^{\frac{1}{3}}\left\{1-\frac{1-s}{1+s}\omega\right\}^{-\frac{1}{3}}.$$

The first factor (so defined that $-\frac{1}{3}\pi \leqslant \arg(-\omega)^{\frac{1}{3}} \leqslant \frac{1}{3}\pi$) has a simple Fourier series expansion on $|\omega| = 1$, while the second and third factors may be expanded as power series in $\omega^{-1}$ and $\omega$ respectively, convergent on $|\omega| = 1$. We may follow a similar procedure with the fractional-power terms in $\mathscr{A}$ and $\mathscr{B}$. Then the left-hand side of (3.7) may be expressed as a series in positive and negative powers of $\omega$, or equivalently as a Fourier series in $\arg\omega$.

To solve (3.7) we then truncate the series (3.4) after $N-1$ terms, and equate coefficients of the first $N$ Fourier components. This gives $N$ equations for the $N$ unknowns $a_0, a_1, \ldots, a_{N-2}$ and $c^2$. This procedure was programmed in FORTRAN IV using the routine C05PAF of the Nottingham Algorithms Group Library to find a solution of

| $m$ | $a_m$ | $m$ | $a_m$ |
|---|---|---|---|
| 0 | 1·4676 | 13 | −0·0029 |
| 1 | −0·1098 | 14 | 0·0024 |
| 2 | 0·0616 | 15 | −0·0018 |
| 3 | −0·0452 | 16 | 0·0015 |
| 4 | 0·0315 | 17 | −0·0011 |
| 5 | −0·0239 | 18 | 0·0010 |
| 6 | 0·0176 | 19 | −0·0007 |
| 7 | −0·0136 | 20 | 0·0006 |
| 8 | 0·0103 | 21 | −0·0004 |
| 9 | −0·0080 | 22 | 0·0004 |
| 10 | 0·0062 | 23 | −0·0002 |
| 11 | −0·0048 | 24 | 0·0003 |
| 12 | 0·0038 | 25 | −0·0001 |
| | | 26 | 0·0002 |

TABLE 2. Coefficients $a_m$ in the series for the lowest-order outer solution when $s = 10$ and $N = 40$ or 60.

the $N \times N$ system of nonlinear algebraic equations, and was run on the IBM 370/165 at Cambridge University. Although there was no assurance that such a system would have a unique solution, in practice the routine always converged rapidly to a particular set of values for the unknowns independently of the starting conditions.

As a check, the values of $z$ and $dz/d\chi$ were computed from (3.4) and (3.6) at points on the free surface, and were substituted in the left-hand side of the surface condition (3.1). Table 1 shows the maximum departure of the value from unity, together with

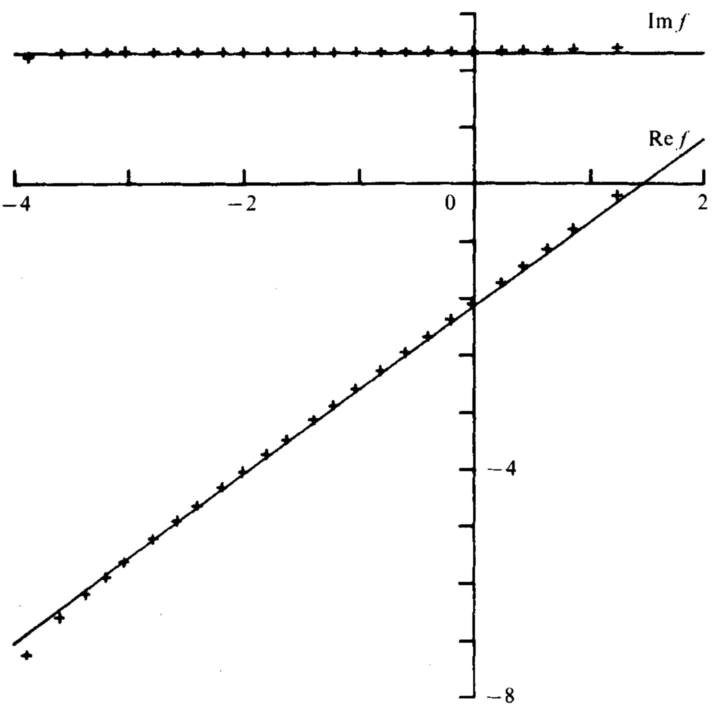



FIGURE 4. The real and imaginary parts of $f = \ln\{z-(\frac{4}{3}i\chi)^{\frac{3}{4}}\}$ on the free surface $\psi = 0$, plotted as functions of $\phi$.

the value of $c^2$ obtained, for a number of runs, in which for three different values of the parameter $s$ the number of terms in the series was progressively increased. The error decreased steadily with an increasing number of terms, and $c^2$ appears to converge to a value of 1·1931. The best rate of convergence was with $s = 10$. Table 2 shows the values of the first 26 coefficients in this case.

In figure 4, $f = \ln\{z-(\frac{4}{3}i\chi)^{\frac{3}{4}}\}$ has been plotted against $\ln\phi$ for points on the free surface. If the asymptotic form as $\chi\to 0$ is to be that of (2.10) we expect

$$\begin{aligned} f &\sim \ln B - \tfrac{1}{2}(\tfrac{1}{3}+\lambda_1)\pi i - (\tfrac{1}{3}+\lambda_1)\ln\phi \\ &= \ln B + 2{\cdot}31i + 1{\cdot}47\ln\phi. \end{aligned}$$

The computed points follow this form very closely, with the exception of those for very small $\phi$, where the power series in $\omega$ has insufficient resolution. The value of $B$ is found to be 0·131.

## 4. The outer solution: first correction

We saw in §2 that the first correction to the outer solution is a function of $\chi$ which has the asymptotic form (2.18) as $\chi\to 0$ and which must be analytic in the flow region and symmetric about $\phi = 0$. We write

$$z = z_0(i\chi) + \epsilon^{3+3i\mu}(1-e^{-i\chi/c})^{-i\mu-\frac{1}{3}}(b_0+b_1\omega+\dots) + \text{complex-conjugate function of } \omega, \quad (4.1)$$

where the $b_n$ are complex, and $z_0(i\chi)$ is the lowest-order outer solution found in §3. We must also allow for a similar correction in the phase speed:

$$c^2 = c_0^2 + \epsilon^{3+3i\mu}c_1^2 + \epsilon^{3-3i\mu}c_1^{*2}. \quad (4.2)$$

To ensure that (2.18) does indeed represent the limiting form of the correction terms in (4.1) as $\chi\to 0$ we impose the condition

$$b_0+b_1+b_2+\dots = c^{-i\mu-\frac{1}{3}}A. \quad (4.3)$$

This form for $z$ is then substituted in the free-surface condition (3.1) and terms of order $\epsilon^3$ are collected. These are of two types: those involving $\epsilon^{3+3i\mu}$ and the rest, multiplied by $\epsilon^{3-3i\mu}$. Since the free-surface condition must be satisfied for all values of $\epsilon$ we equate coefficients of each of these terms separately. From (4.1) we have

$$\begin{aligned} 2\,\mathrm{Re}\,z = {} & (1-e^{-i\chi/c})^{\frac{2}{3}}\mathscr{A} + \epsilon^{3+3i\mu}(1-e^{-i\chi})^{-\frac{1}{3}}\mathscr{C} \\ & + \epsilon^{3-3i\mu}(1-e^{-i\chi/c})^{-\frac{1}{3}}\overline{\mathscr{C}} + \text{complex-conjugate terms}, \end{aligned} \quad (4.4)$$

where

$$\mathscr{C}(\omega) = \left\{\frac{2(1-\omega)}{(1+s)-(1-s)\,\omega}\right\}^{-i\mu}(b_0+b_1\omega+\dots),$$
$$\overline{\mathscr{C}}(\omega) = \{\mathscr{C}(\omega^*)\}^*,$$

and $\mathscr{A}$ is as defined in §3. On differentiating (4.1) we have

$$\begin{aligned} -ic\,dz/d\chi = {} & (1-e^{-i\chi/c})^{-\frac{1}{3}}\{(1-s)\,\omega-(1+s)\}^{-1}\mathscr{B} \\ & + \epsilon^{3+3i\mu}(1-e^{-i\chi/c})^{-\frac{4}{3}}\{(1-s)\,\omega-(1+s)\}^{-1}\mathscr{D} \\ & + \epsilon^{3-3i\mu}(1-e^{-i\chi/c})^{-\frac{4}{3}}\{(1-s)\,\omega-(1+s)\}^{-1}\overline{\mathscr{D}}, \end{aligned} \quad (4.5)$$

| $m$ | $b_m$ | | $m$ | $b_m$ | |
|---|---|---|---|---|---|
| 0 | 0·1199 | −0·1454 $i$ | 10 | 0·0101 | −0·0046 $i$ |
| 1 | 0·0390 | 0·1167 $i$ | 11 | −0·0083 | 0·0025 $i$ |
| 2 | 0·0243 | −0·0945 $i$ | 12 | 0·0065 | −0·0022 $i$ |
| 3 | −0·0292 | 0·0574 $i$ | 13 | −0·0054 | 0·0010 $i$ |
| 4 | 0·0304 | −0·0421 $i$ | 14 | 0·0042 | −0·0011 $i$ |
| 5 | −0·0264 | 0·0265 $i$ | 15 | −0·0035 | 0·0003 $i$ |
| 6 | 0·0228 | −0·0198 $i$ | 16 | 0·0026 | −0·0005 $i$ |
| 7 | −0·0188 | 0·0124 $i$ | 17 | −0·0023 | 0·0000 $i$ |
| 8 | 0·0155 | −0·0096 $i$ | 18 | 0·0017 | −0·0002 $i$ |
| 9 | −0·0126 | 0·0057 $i$ | 19 | −0·0015 | −0·0001 $i$ |

TABLE 3. Coefficients $b_m$ for the first 20 terms in the series for the correction to the outer solution, when $s = 10$ and $N = 40$ or 60.

where
$$\begin{aligned}\mathscr{D}(\omega) &= \left\{\frac{2(1-\omega)}{(1+s)-(1-s)\,\omega}\right\}^{-i\mu} \\ &\quad\times [\{(1-s)-(1+s)\,\omega\}\,(-\tfrac{1}{3}-i\mu)\,(b_0+b_1\omega+\dots) \\ &\quad+(2s)^{-1}\{(1-s)-(1+s)\,\omega\}\{(1-s)\,\omega-(1+s)\}\,(1-\omega)\,(b_1+2b_2\omega+\dots)], \\ \overline{\mathscr{D}}(\omega) &= \{\mathscr{D}(\omega^*)\}^*,\end{aligned}$$
and $\mathscr{B}$ is as defined in §3.

The terms in $\epsilon^{3+3i\mu}$ in (3.1) then give
$$\begin{aligned}\mathscr{G}(\omega)+\mathscr{G}(\omega^{-1}) &= \frac{2}{\omega}\left\{\frac{1-\omega}{(1+s)-(1-s)\,\omega}\right\}^{\frac{1}{3}}\left\{\frac{1-\omega^{-1}}{(1+s)-(1-s)\,\omega^{-1}}\right\}^{\frac{1}{3}} \\ &\quad\times c_1^2\{(1-s)\,\omega-(1+s)\}\{(1-s)-(1+s)\,\omega\},\end{aligned}\tag{4.6}$$
where
$$\begin{aligned}\mathscr{G}(\omega) &\equiv (-\omega)^{-\frac{1}{6}}\left\{\frac{(1+s)-(1-s)\,\omega}{(1+s)-(1-s)\,\omega^{-1}}\right\}^{\frac{1}{6}}\{\mathscr{A}(\omega)\,\mathscr{D}(\omega)+\mathscr{B}(\omega)\,\mathscr{C}(\omega)\}\,\mathscr{B}(\omega^{-1}) \\ &\quad+(-\omega)^{\frac{5}{6}}\left\{\frac{(1+s)-(1-s)\,\omega}{(1+s)-(1-s)\,\omega^{-1}}\right\}^{-\frac{5}{6}}\mathscr{A}(\omega)\,\mathscr{B}(\omega)\,\mathscr{D}(\omega^{-1}).\end{aligned}$$

Consideration of the terms in $\epsilon^{3-3i\mu}$ leads to the complex conjugate of (4.6).

By the same methods as were used in §3 both the right- and the left-hand side of (4.6) may be expressed as a Fourier series in $\arg\omega$. Truncating the series (4.1) after $N-1$ terms, and equating the first $N-1$ Fourier components of (4.6), we have, together with (4.3), $N$ complex linear equations for the $N$ complex unknowns $b_0, b_1, \dots, b_{N-2}$ and $c_1^2$.

## 5. Results: variation of wave properties

As $s = 10$ gave the best convergence for the lowest-order outer solution we take this value in solving the linear equations of §4 for the first correction, with $N = 60$ as before. The series $\mathscr{A}$ and $\mathscr{B}$ have already beeen determined by the computations of §3. Table 3 shows the values obtained for the coefficients $b_0, b_1, \dots, b_{19}$. We find also
$$c_1^2 = 0{\cdot}5887e^{-0{\cdot}9233i}. \tag{5.1}$$

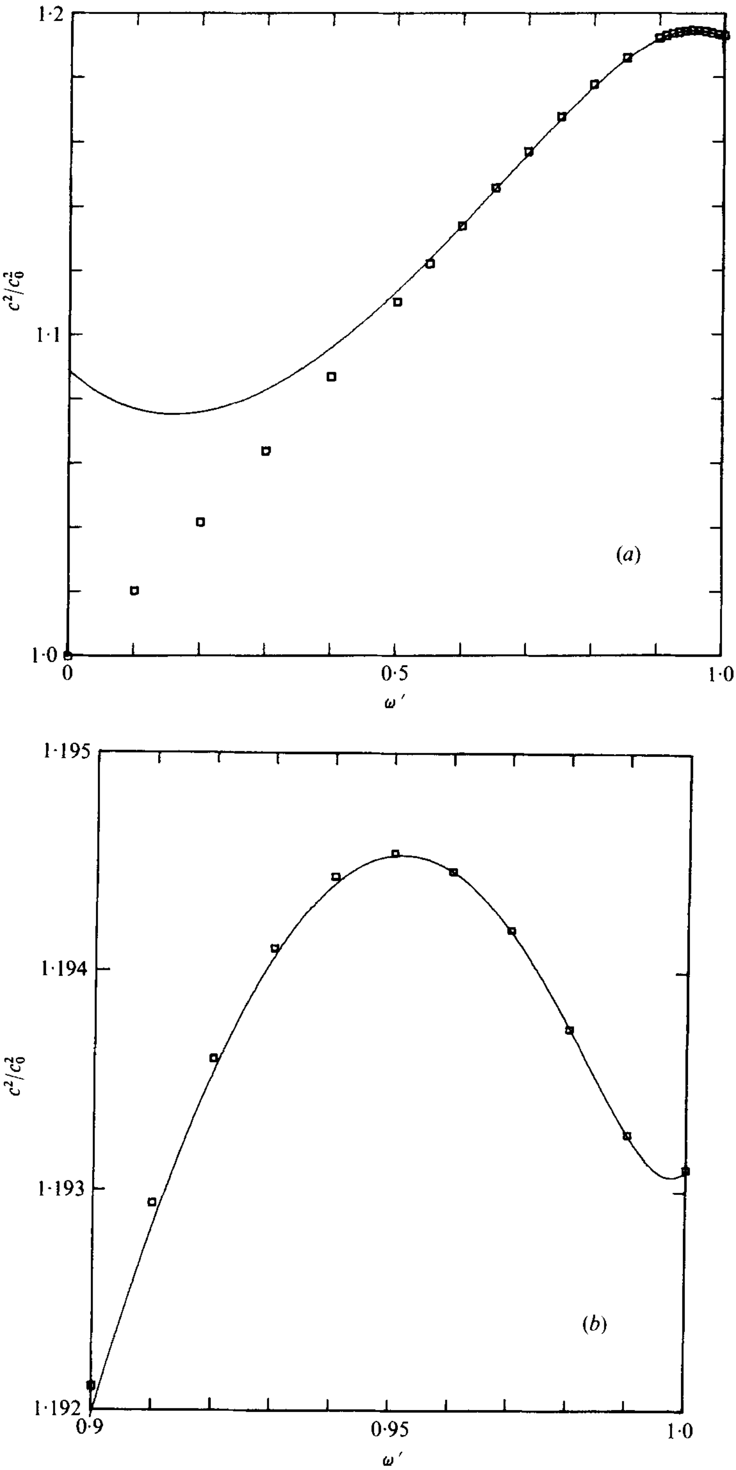
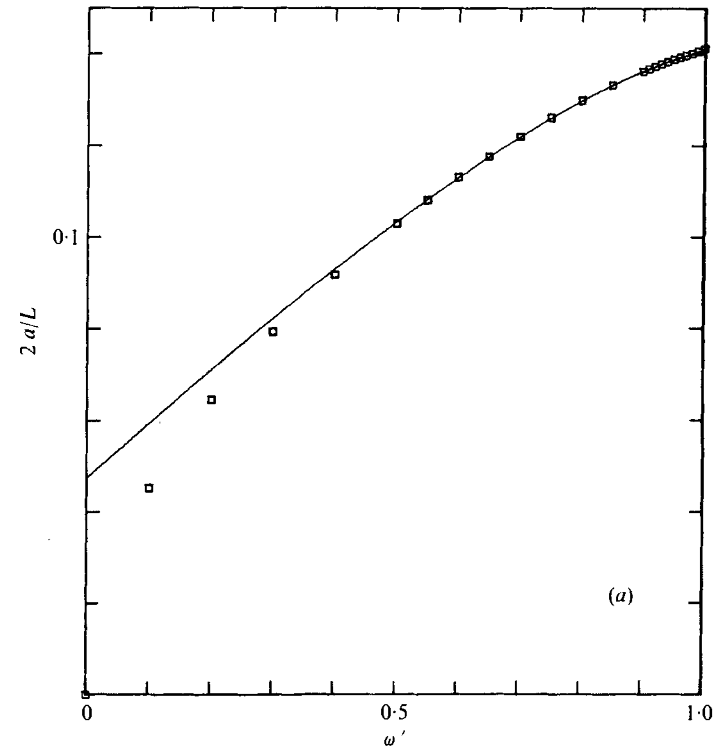



FIGURE 5. (a) The square of the phase speed $c$ in a deep-water wave, shown as a function of $\omega'$. The squares represent values obtained by Padé summation of high-order series. The curve represents the asymptotic formula (5.3). (b) Enlargement of (a) at high values of $\omega'$.



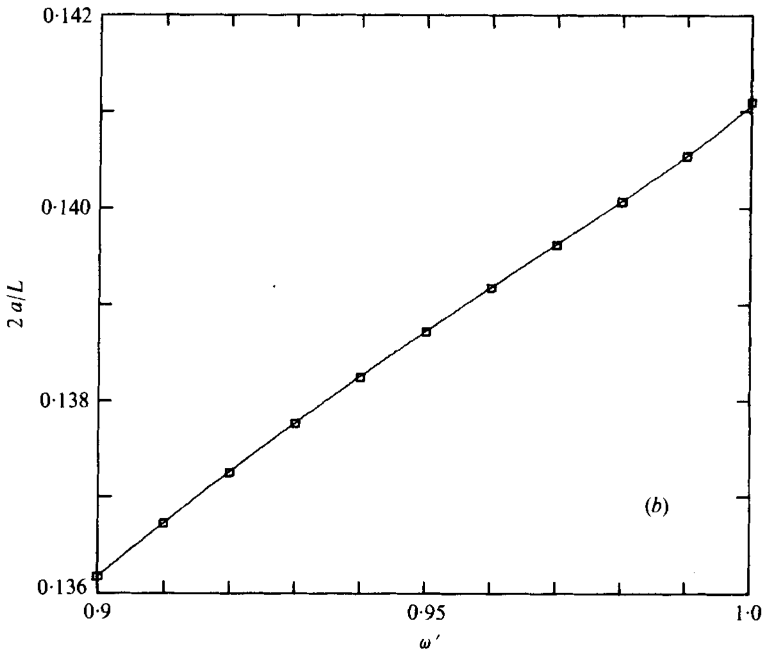




FIGURE 6. (*a*) The wave steepness $2a/L$ as a function of $\omega'$. The squares represent values obtained by Padé summation of high-order series. The curve represents the asymptotic formula (5.5). (*b*) Enlargement of (*a*) at high values of $\omega'$.

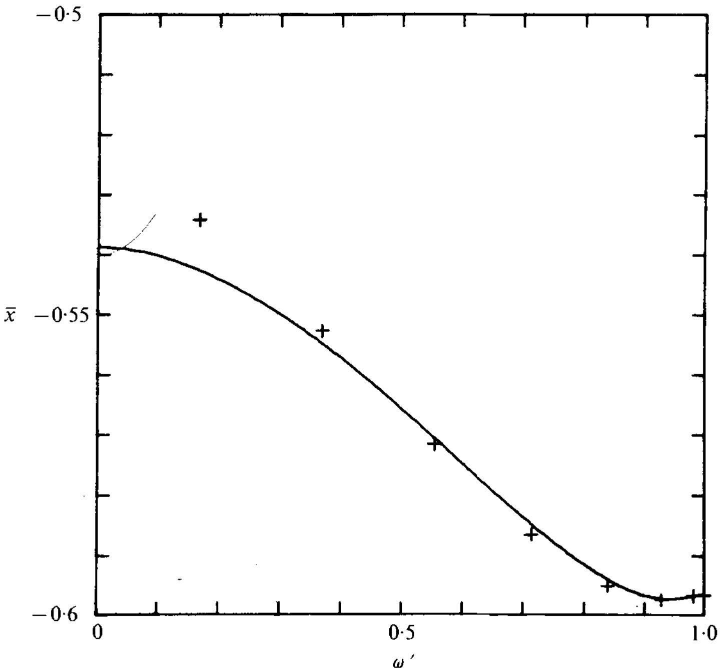



FIGURE 7. The mean level as a function of $\omega'$. The crosses are derived from a numerical integration of the (asymptotic) surface profile. The curve represents the asymptotic formula (5.6).

This yields an expression for $c^2$ in dimensional terms, to order $\epsilon^3$,

$$c^2 = (g/k)\{1{\cdot}1931 - 1{\cdot}18\epsilon^3 \cos(2{\cdot}143 \ln \epsilon + 2{\cdot}22)\}. \tag{5.2}$$

In figures 5(*a*) and (*b*) this has been plotted as a function of $\omega' = 1 - 2\epsilon^2 q_t^2/c^2$, where $q_t$ is the fluid velocity in the wave trough, and compared with the results of Longuet-Higgins (1975), derived from a small amplitude expansion. The agreement is remarkably good, the two-term expansion for $c^2$ apparently being a good representation for values of $\omega'$ down to about 0·6. In particular the existence of a maximum in the wave speed at $\omega' \simeq 0{\cdot}95$ is confirmed. This analysis also indicates that the wave speed passes through an infinite succession of maxima and minima as $\omega'$ approaches 1.

The wave steepness is given by

$$a/\pi = (x_{\text{trough}} - x_{\text{crest}})/2\pi, \tag{5.3}$$

where we know from (4.1) that

$$x_{\text{trough}} = \operatorname{Re}\{z_0(i\pi c) + \epsilon^{3+3i\mu}2^{-i\mu-\frac{1}{2}}(b_0 - b_1 + b_2 - \ldots) + \epsilon^{3-3i\mu}2^{i\mu-\frac{1}{2}}(b_0^* - b_1^* + b_2^* - \ldots)\}$$

and $x_{\text{crest}} = \epsilon^2$, from the definition (2.1) of $\epsilon$ together with the constant-pressure condition (3.1). Equation (5.3) then becomes

$$a/\pi = 0{\cdot}14107 - 0{\cdot}50\pi^{-1}\epsilon^2 + 0{\cdot}160\epsilon^3 \cos(2{\cdot}143 \ln \epsilon - 1{\cdot}54), \tag{5.4}$$

which is shown in figures 6(*a*) and (*b*) as a function of $\omega'$, again compared with the results of Longuet-Higgins (1975). The wave steepness is a monotonic function of $\omega'$, and also of $(x_{\text{crest}} - x_{\text{trough}})$, the oscillatory behaviour of $x_{\text{trough}}$ being insufficient to overcome the higher-order monotonic dependence of $x_{\text{crest}}$. The limiting, sharp-crested wave is therefore the highest, although not the fastest.

The origin of co-ordinates in the analysis so far has been taken at a point above the wave crest such that the constant in the Bernoulli free-surface condition vanishes. For practical purposes it is sometimes more convenient to take the origin at the mean surface level of the wave, and so we now calculate this level, which is given by

$$\begin{aligned}\bar{x} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dy \\ &= \tfrac{1}{2}x_{\text{trough}} - \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{\pi c} iz^*\frac{dz}{d\chi}d\chi\right),\end{aligned} \tag{5.5}$$

upon integrating by parts. The lowest-order value of $\bar{x}$ is then obtained by substitution of the lowest-order outer solution of § 3 in (5.5). To obtain the first correction to $\bar{x}$ we may substitute the corrected form (4.1) for $z$ in (5.5) and collect terms of order $\epsilon^3$, neglecting terms of order $\epsilon^6$. Equation (4.1) is valid only in the zones II and III defined in § 2; however, the contribution it makes in zone I to the integral of (5.5) is of order

$$\epsilon^3\int_0^{\epsilon^3}\chi^{-\frac{2}{3}}\,d\chi = O(\epsilon^4)$$

while the correction contribution in this zone, obtained from the inner solution, would be of order

$$\int_0^{\epsilon^3}\epsilon\,d\chi = O(\epsilon^4).$$

Neglecting these terms we obtain

$$\bar{x} = k^{-1}\{0{\cdot}59654 - 0{\cdot}588\epsilon^3\cos(2{\cdot}143\ln\epsilon + 2{\cdot}22)\}. \tag{5.6}$$

This expression has been plotted as a function of $\omega'$ in figure 7. The points marked for comparison were obtained directly from the computed form of the free surface for some values of $\epsilon$.

## 6. Integral properties of the wave

We consider next the mean potential energy $V$ per unit area, which is given by

$$\begin{aligned}2\pi V &= \int_{-\pi}^{\pi}\tfrac{1}{2}(x-\bar{x})^2\,dy \\ &= \int_0^{\pi} x^2\,dy - \pi\bar{x}^2 \\ &= \tfrac{1}{3}\pi x^2_{\text{trough}} - \pi\bar{x}^2 - \tfrac{1}{3}\operatorname{Re}\int_0^{\pi c}\left(iz\bar{z}\frac{dz}{d\chi} - iz^2\frac{d\bar{z}}{d\chi}\right)d\chi\end{aligned} \tag{6.1}$$

upon integration by parts. Again we may find the lowest-order value and the first correction by substituting (4.1) in (6.1) and retaining terms up to order $\epsilon^3$, the error arising from the use of the wrong form of the integrand in zone I being of order

$$\epsilon^3\int_0^{\epsilon^3}d\chi \sim \epsilon^6.$$

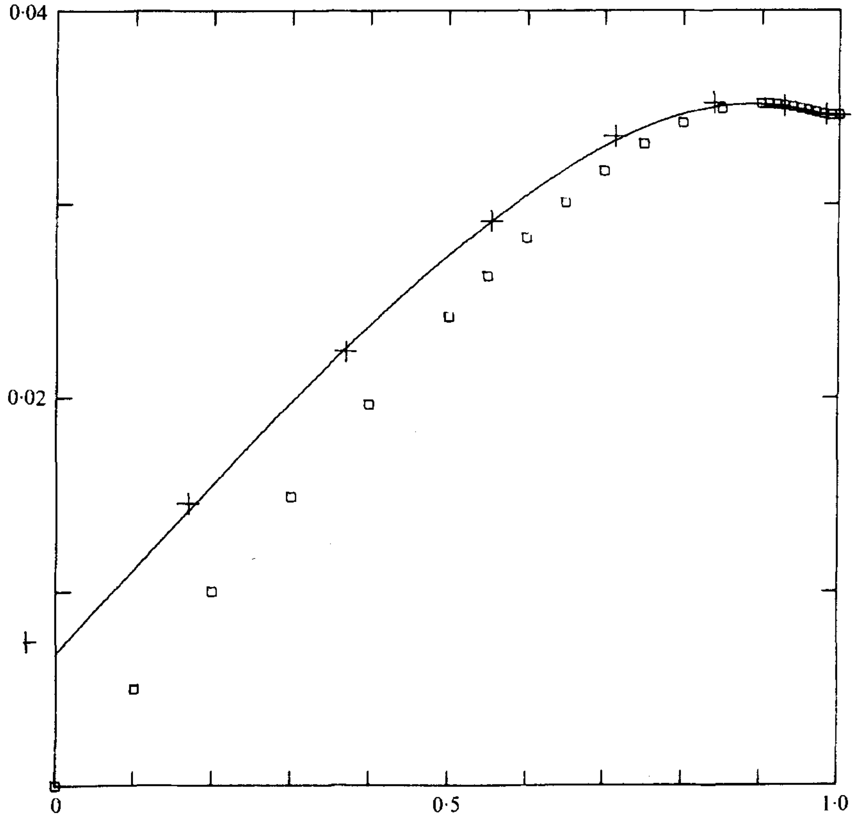



FIGURE 8. The dimensionless potential energy $V$ in a deep-water wave as a function of $\omega'$. The squares represent Padé sums; the curve represents the asymptotic formula (6.4). The crosses are derived from integration of the asymptotic profile.

Figure 8 shows the resulting expression

$$V = (g/k^2)\{0{\cdot}03457 - 0{\cdot}169\epsilon^3 \cos(2{\cdot}143 \ln \epsilon + 1{\cdot}49)\} \tag{6.2}$$

as a function of $\omega'$ compared with the results of Longuet-Higgins (1975).

The mean impulse I per unit area is defined by

$$I = \frac{1}{\pi}\int_0^\pi \int_x^\infty (v - c)\, dx\, dy, \tag{6.3}$$

where $v$ is the fluid velocity in the $y$ direction. In terms of the stream function $\psi$,

$$I = \frac{1}{\pi}\int_0^\pi \int_x^\infty \left(-\frac{\partial \psi}{\partial x} - c\right) dx\, dy = -\frac{1}{\pi}\int_0^\pi (\psi + cx)_x^\infty\, dy. \tag{6.4}$$

On the surface $\psi = 0$, and as $x \to \infty$

$$\begin{aligned} \psi + cx \to c\{&-1 + (a_0 + a_1\overline{\omega} + a_2\overline{\omega}^2 + \ldots) \\ &+ \epsilon^{3+3i\mu}(b_0 + b_1\overline{\omega} + \ldots) + \epsilon^{3-3i\mu}(b_0^* + b_1^*\overline{\omega} + \ldots)\} \\ = {}& K_0 + K_1\epsilon^{3+3i\mu} + K_1^*\epsilon^{3-3i\mu} \end{aligned}$$

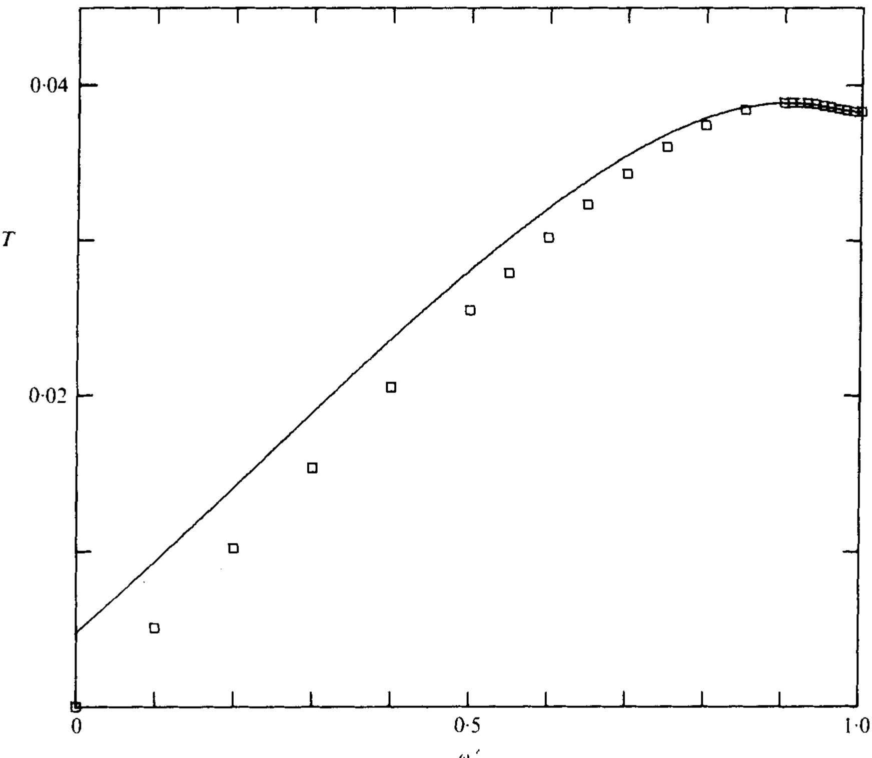



FIGURE 9. The dimensionless kinetic energy $T$ as a function of $\omega'$.

say, from (3.4) and (4.1), where $\overline{\omega} = (1-s)/(1+s)$. Equation (6.4) then becomes

$$I = c\overline{x} - K_0 - K_1 \epsilon^{3+3i\mu} - K_1^* \epsilon^{3-3i\mu}.$$

Using (5.6) we may write, in dimensional terms,

$$I = g^{\frac{1}{2}} k^{-\frac{3}{2}} \{0{\cdot}07011 - 0{\cdot}364\epsilon^3 \cos(2{\cdot}143 \ln \epsilon + 1{\cdot}61)\}. \tag{6.5}$$

This is compared with the results of Longuet-Higgins (1975) in figure 10.

Finally we can obtain an expression for the mean kinetic energy $T$ per unit area from the relationship $2T = cI$ (Levi-Civita 1924). From the above expressions for $c$ and $I$ we find

$$T = gk^{-2} \{0{\cdot}03829 - 0{\cdot}215\epsilon^3 \cos(2{\cdot}143 \ln \epsilon + 1{\cdot}66)\} \tag{6.6}$$

correct to order $\epsilon^3$. Figure 9 shows the corresponding comparison for this quantity.

## 7. Analytic extension of the flow field

A question of some interest is the nature of the singularities of the flow field in a steep gravity wave when this is continued analytically across the free surface. As the wave approaches its limiting form, with the length scale of the crest tending to zero, we expect that there will be singularities near the crest which will move towards the surface, finally coalescing to form the 120° corner singularity of the limiting wave. Grant (1973) has suggested that these singularities, for a wave which has not attained the limiting form, must be of order one-half when the complex co-ordinate $z = x + iy$ is

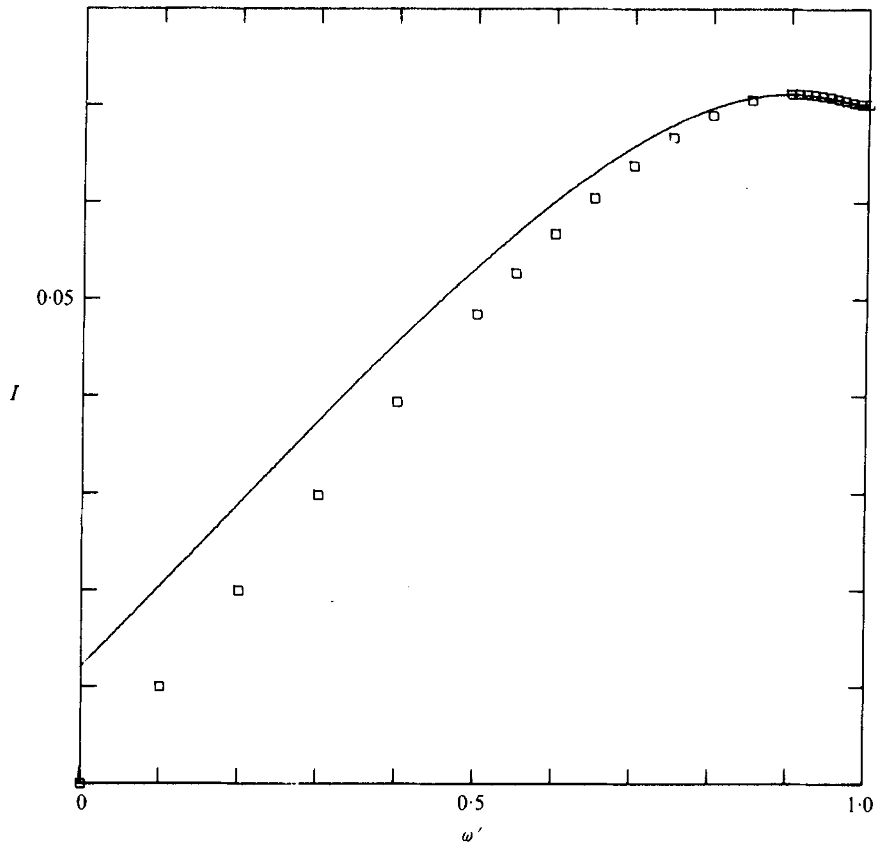



FIGURE 10. The dimensionless wave impulse $I$ (momentum flux) as a function of $\omega'$.

expressed as a function of $\chi$. If these singularities are indeed of order one-half in $\chi$, then $\chi-\chi_0$ will behave locally as $(z-z_0)^2$ so there will be a stagnation point in the flow field with streamlines meeting at right angles.

To investigate the question we analytically continued the flow near the crest across the free surface, first expanding the term $(\delta+i\chi)^{\frac{2}{3}}$ in (6.4) of paper I as a power series in $\omega$ to give

$$z = \hat{z}+z_1\omega+z_2\omega^2+\ldots+z_N\omega^N, \quad N = 59.$$

This power series was then reverted to give $\omega$ as a power series in $(z-\hat{z})$. Finally we computed the [29, 29] Padé approximant to this 60-term series and plotted contours of

$$\psi = \mathrm{Re}\left(\beta\frac{1-\omega}{1+\omega}\right)$$

in the $\beta$ plane. These are shown in figure 11. One of the predicted stagnation points occurs on the real axis at $x/l = -2{\cdot}9$. The regions of intense gradients of $\psi$ at larger negative values of $z$ indicate poles of the Padé approximant, which may represent a branch cut along the $-x$ axis. If this is of the form

$$\chi-\chi_0 = C(z-z_0)^{1/n},$$

where $n$ is an integer, then

$$z-z_0 = C^{-n}(\chi-\chi_0)^n,$$

which is regular in the $\chi$ plane, and so is consistent with Grant's result.

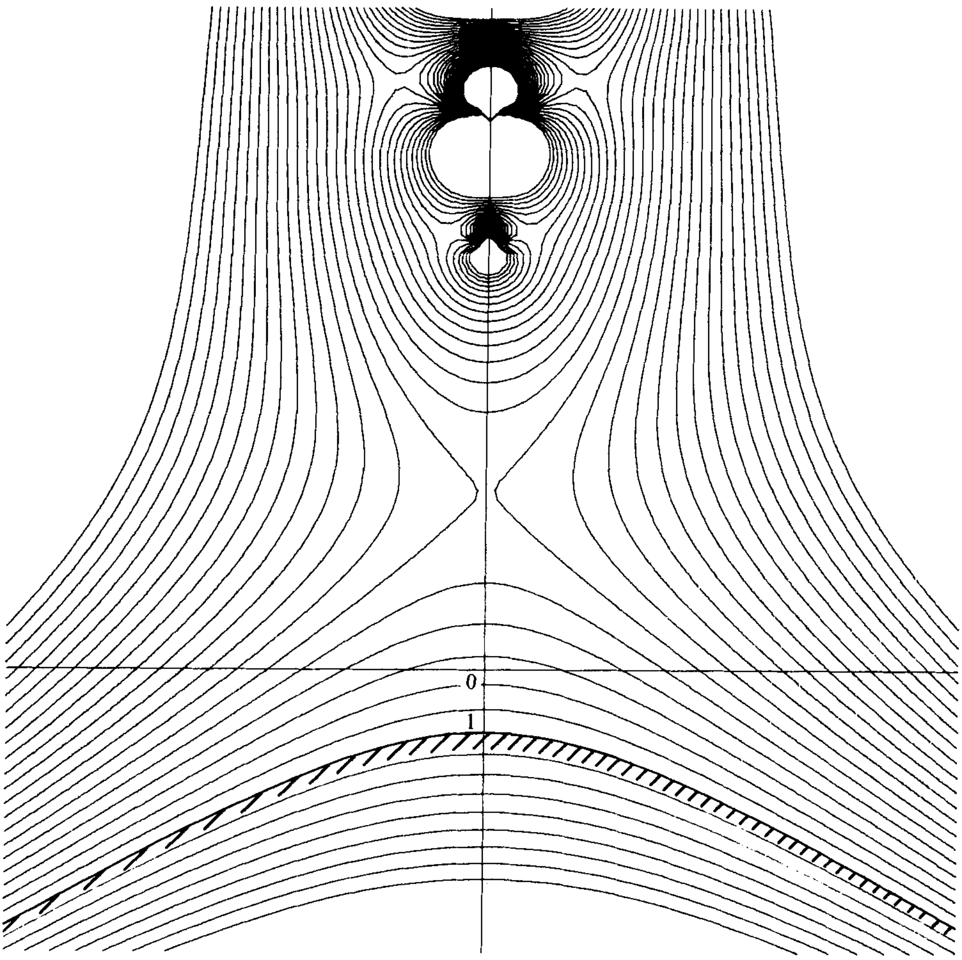



FIGURE 11. Streamlines of the flow near the wave crest, extended analytically across the free surface (shown by hatching). From the solution of paper I.

Figure 11 is constructed on the length scale of the inner solution, as in paper I, that is to say in units of

$$l = q^2/2g.$$

According to the discussion in § 3 above; this scale is of order $\epsilon^2 L$, where $L$ is the overall wavelength. As $\epsilon \to 0$ so the singularities of the inner solution will presumably approach the free surface from above and will ultimately coalesce at the wave crest.

## 8. Conclusion

We have shown how the flow near the crest of a steep gravity wave may be matched to the flow in the main body of the wave by the addition of a correction at order $\epsilon^3$, $\epsilon$ being a small parameter measuring the deviation from the limiting form. The correction also involves imaginary powers of $\epsilon$. The resulting expressions (5.2) for the wave speed, (5.4) for the steepness, (6.2) and (6.6) for the potential and kinetic energies, and (6.5) for the impulse provide simpler and more accurate values of these quantities in steep waves than those available from small amplitude expansions. Confirmation is obtained of the existence of maxima of the wave speed, energies and impulse as functions of the amplitude.

In a subsequent paper it will be shown how the present method may be applied to the computation of steep solitary waves.

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